Product and quotient rules

Introduction

As their names suggest, the **product rule** and the **quotient rule** are used to differentiate products of functions and quotients of functions. This leaflet explains how.

1.The product rule

It is appropriate to use this rule when you want to differentiate two functions which are multiplied together. For example

$$y = e^x \sin x \qquad \text{is a product of the functions } e^x \text{ and } \sin x$$

In the rule which follows we let u stand for the first of the functions and v stand for the second.

> If u and v are functions of x, then
>
> $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

If $y = 7xe^{2x}$ find $\dfrac{dy}{dx}$.

Solution

Comparing the given function with the product rule we let

$$u = 7x, \qquad v = e^{2x}$$

It follows that

$$\frac{du}{dx} = 7, \qquad \text{and} \qquad \frac{dv}{dx} = 2e^{2x}$$

Thus, using the product rule,

$$\frac{d}{dx}(7xe^{2x}) = 7x(2e^{2x}) + e^{2x}(7) = 7e^{2x}(2x+1)$$

2. The quotient rule

It is appropriate to use this rule when you want to differentiate a quotient of two functions, that is, one function divided by another. For example

$$y = \frac{\mathrm{e}^x}{\sin x} \qquad \text{is a quotient of the functions } \mathrm{e}^x \text{ and } \sin x$$

In the rule which follows we let u stand for the function in the numerator and v stand for the function in the denominator.

If u and v are functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{du}{\mathrm{d}x} - u\frac{dv}{\mathrm{d}x}}{v^2}$$

Example

If $y = \dfrac{\sin x}{3x^2}$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Comparing the given function with the quotient rule we let

$$u = \sin x, \qquad \text{and} \qquad v = 3x^2$$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 6x$$

Applying the quotient rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2\cos x - \sin x\,(6x)}{9x^4} = \frac{3x(x\cos x - 2\sin x)}{9x^4} = \frac{x\cos x - 2\sin x}{3x^3}$$

Exercises

Choose an appropriate rule in each case to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

1. $y = x^2\sin x$
2. $y = \mathrm{e}^x\cos x$
3. $y = \frac{\mathrm{e}^x}{x^2+1}$
4. $y = \frac{x^2+1}{\mathrm{e}^x}$
5. $y = 7x\log_\mathrm{e} x$
6. $y = \frac{x-1}{\sin 2x}$

Answers

1. $x^2\cos x + 2x\sin x$ 2. $-\mathrm{e}^x\sin x + \mathrm{e}^x\cos x = \mathrm{e}^x(\cos x - \sin x)$ 3. $\frac{\mathrm{e}^x(x^2-2x+1)}{(x^2+1)^2}$

4. $\frac{2x-x^2-1}{\mathrm{e}^x}$, 5. $7(1+\log_\mathrm{e} x)$, 6. $\frac{\sin 2x - 2(x-1)\cos 2x}{\sin^2 2x}$.

